A formal description of discrete event dynamic systems including perturbation analysis

Bettonvil, B.W.M.

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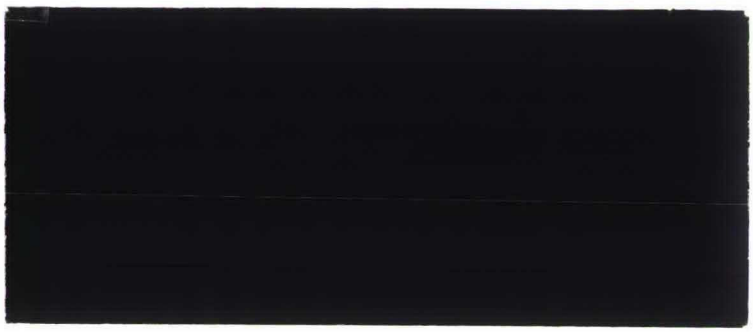
UNIVERSITEIT BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS



DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

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Bert Bettonvil

Bert Bettonvil
Catholic University Tilburg/Technical University Eindhoven
P O Box 90152
5000 LE Tilburg
Netherlands

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A FORMAL DESCRIPTION OF DISCRETE EVENT DYNAMIC SYSTEMS INCLUDING PERTURBATION ANALYSIS

Bert Bettonvil
Catholic University Tilburg/Technical University Eindhoven
P O Box 90152
5000 LE Tilburg
Netherlands

We focus on the behaviour of simulation outcomes of discrete event dynamic systems. This behaviour is characterized by several kinds of Performance Measures. We arrive at a description of a wide class of discrete event dynamic systems. Within this context we define Perturbation Analysis, a technique to quantify the dependence of simulation outcomes on changes in parameter inputs.

Subject classification: 761 Discrete Event Dynamic Systems and Perturbation Analysis.

A simulation practitioner is not interested in a detailed description of all stages a simulation goes through; he or she is merely interested in one or a few global measures of its performance. This paper is inspired by the work of Ho and his group of researchers at Harvard University concerning Perturbation Analysis (P/A) (see, e.g. Ho (1983), Suri (1984), Ho *et. al.* (1984), including further references). Although the technique of P/A itself has met essential criticism from Heidelberger (1986), the fundaments on which it is based may still prove worthwile.

These fundaments are formed by a general and cohesive description of a commonly used kind of simulation, namely Discrete Event Dynamic System (DEDS) simulation, in which a real world system is modelled as a series of

events; between two consecutive events the system does not alter. Simulation-programs for DEDS "jump" from event to successor-event, each time adjusting a clock, and updating performance measures. The latter are used to compute the final performance measures, the user is really interested in.

In section 1 a detailed description of a DEDS is given; in section 2 we introduce deratives of random variables; in section 3 P/A is described as an example of the description in section 1; section 4 contains a discussion.

1. System Description

A DEDS can be described as a series of events. Between two events, the system does not change; it is in some state. The points of time at which the events take place (or, equivalently, the system changes from one state to another) are called the eventtimes. The behaviour of a system is completely specified by the initial state, and an enumeration of the actual events and eventtimes.

> Example: In an M/M/1-queue there are two types of events: arrival and departure; and a countable number of states: in state 0 the system is empty; in state 1 a customer is served, no one is waiting; in state 2 a customer is served, another one is waiting; etc. A simulation experiment results in an initial state, a sequence of arrivals and departures, and their eventtimese. In this case, we might as well enumerate the states instead of the events; however, this is not always so.

In general, an enumeration of the actual states is not sufficient to describe the behaviour of the system, as the following example shows.

> Example. Suppose we have two parallel servers, that work at different costs. Both are busy, and one customer is waiting. In the next state, both devices are busy, no customer is waiting. We do not know what happened in between.

Not every sequence of events is possible (e.g. no customer can leave an empty system); the state of the system determines which events can possibly happen next. By X we denote the set of system-states. Given any $x \in X$ there is a finite set of events that can take place next. This set is denoted by $E(x)$, the exit-states of x.

> Example. In the M/M/1-queue we have $X=\{0,1,2,\ldots\}$. The set of all events is $\{A,D\}$, with arrival A and departure D. If the system is in state 0, only A can be the next event; in any other state, both A and D are possible. So $E(0)=\{A\}$; $E(1)=E(2)=\ldots=\{A,D\}$.

To ascertain that we can check whether a given sequence of actual events is indeed possible, we need the sequence of states. Suppose the system started in state x_0, and events $e_1, e_2, \ldots, e_{k-1}$ occured. To make sure that a certain event e can be the next event, e must be an element of $E(x_{k-1})$, in which x_{k-1} is the present state. So x_{k-1} must be known. This can be achieved by requiring: a state and an event (which is possible in that state) together completely determine the next state.

When evaluating a simulation, we do not want an exhausive description of every step it went through, we want its behaviour to be characterized by one or a few performance measures (PM's). A PM is a function of the states and events the system goes through during the simulation, and the duration of these states.

> Example. Suppose we want to know the mean number of customers waiting to be served, and the mean waiting time per served customer, for the M/M/1-queue. We start in an arbitrary state and stop the simulation when N customers have been treated. The mean number of customers is $\Sigma_{i=0}^{\infty} i p_i$ in which p_i is the probability that i customers are waiting. We estimate p_i by the time during which i customers are waiting divided by the total simulation time. We may as well compute the total waiting time and divide this quantity by the total simulation time.

The mean waiting time per served customer can be computed as the total waiting time divided by the total number of customers served. So we need total waiting time, total simulation time, and total number of customers served. These quantities can easily be computed by accumulation at each event.

Suppose the simulation consists of the series of states $x_0, x_1, x_2, \ldots$ with $x_i \in X$ $(i=0,1,2,\ldots)$. The actual events are $e_1, e_2, \ldots$ that take place at points of time $\Theta_1, \Theta_2, \ldots$. In other words, the simulation starts in the prespecified state x_0 at time $\Theta_0=0$. At Θ_1 event e_1 takes place $(e_1 \in E(x_0))$ and the next state is x_1, etc.

Let the k^{th} <u>interval</u> run from Θ_{k-1} to Θ_k $(k=1,2,\ldots)$. During this interval the system is in state x_{k-1}; the interval ends with e_k. Let each state $x \in X$ correspond with a number $f_s(x)$, and each event $e \in E = \bigcup_{x \in X} E(x)$ correspond with a number $f_o(e)$. A <u>basic PM</u> in the k^{th} interval y_k^b is defined as

$$y_k^b = (\Theta_k - \Theta_{k-1}) f_s(x_{k-1}) + f_o(e_k). \qquad (1.1)$$

A <u>simple PM</u> over the intervals K_1 through K_2 $(K_1 \leq K_2)$ is defined as

$$y_{K_1,K_2} = \frac{1}{K_2 - K_1 + 1} \sum_{k=K_1}^{K_2} y_k^b. \qquad (1.2)$$

Note that a simple PM is defined as the mean of basic PM's. We might as well have defined it as the sum of basic PM's, were it not that, in order to be able to make statements about populations, we want to allow for $K_2 \to \infty$.

We can have several functions f_s and f_o, say $f_s^{(1)}, f_o^{(1)}, f_s^{(2)}, f_o^{(2)}, \ldots, f_s^{(n)}, f_o^{(n)}$. These give n different basic PM's $y_s^{b(1)}, \ldots, y_s^{b(n)}$ and n different simple PM's $y^{(1)}_{K_1^{(1)}, K_2^{(1)}}, \ldots, y^{(n)}_{K_1^{(n)}, K_2^{(n)}}$. A <u>general PM</u> is a function of simple PM's.

Example: To compute the mean number of customers waiting we need the total waiting time and the total simulation time. The waiting time in the k^{th} interval is $(\Theta_k - \Theta_{k-1})\max\{x_{k-1}-1, 0\}$, so we take $f_s^{(1)}(x) = \max\{x-1, 0\}$ and $f_o^{(1)}(e) = 0$ for all e. Then the following quantity is K^{-1} times the total waiting time:

$$y^{(1)}_{1,K} = \frac{1}{K} \sum_{k=1}^{K} y_k^{b(1)} = \frac{1}{K} \sum_{k=1}^{K} (\Theta_k - \Theta_{k-1})\max\{x_{k-1}-1, 0\}.$$

The duration of the k^{th} interval is $\Theta_k - \Theta_{k-1}$. So we take $f_s^{(2)}(x) = 1$ for all x and $f_o^{(2)}(e) = 0$ for all e. Then

$$y^{(2)}_{1,K} = \frac{1}{K} \sum_{k=1}^{K} y_k^{b(2)} = \frac{1}{K} \sum_{k=1}^{K} (\Theta_k - \Theta_{k-1}) = \frac{1}{K}(\Theta_K - \Theta_0)$$

is K^{-1} times the total simulation time. Now $y^{(1)}_{1,K} / y^{(2)}_{1,K}$ is the mean number of customers waiting.

To compute the mean waiting time per served customer we need the total waiting time, that is $Ky^{(1)}_{1,K}$, and the total number of customers served. The number of customers served in the k^{th} interval is 0 if the interval ends with an arrival; it is 1 if a departure terminates the interval. So we take

$$f_s^{(3)}(x) = 0 \text{ for all } x \text{ and } f_o^{(3)}(A) = 0 \; ; \quad f_o^{(3)}(D) = 1.$$

Then

$$y^{(3)}_{1,K} = \frac{1}{K} \sum_{k=1}^{K} y^{b(3)}_{k} = \frac{1}{K} \sum_{k=1}^{K} f^{(3)}_{o}(e_k) = K^{-1} N_D$$

in which N_D is the number of departures in the first K intervals.

Now $y^{(1)}_{1,K} / y^{(3)}_{1,K}$ is the mean waiting time per served customer.

A PM is a function of the states and events the system goes through during the simulation, and the duration of the states. It is possible to define basic PM's in such a way that any PM is a function of basic PM's, and so it is a function of simple PM's. But a function of simple PM's is what we called a general PM. So a general PM is indeed general, i.e. it covers the concept of a PM. In the appendix a formal proof for this statement is given. From this proof a general method for determining the functions f_s and f_o, given a specific PM, is directly deductible.

We saw that, when the system is in state x_{k-1}, there is a finite set of events that can take place next. We called this set $E(x_{k-1})$.

Example. In an empty queueing system the next event is an arrival. When the system is not empty, the event is either an arrival or a departure. Which event will actually happen, depends on the time at which the two possible events are scheduled. In many situations (the M/M/1-queue is one of these) the fact that one event happens, does not have any influence on the points of time at which the other ones are scheduled. However, we do not want to impose restrictions like this on the systems in general.

Associated with all events $e \in E(x_{k-1})$ is a point of time $t_k(e)$, i.e. the time at which e will happen, if nothing else happens before. We leave open the possibility that e won't happen at all, due to some other event.

Example. Now we need some other example than the M/M/1-queue. Suppose we have a closed queueing network, in which two kinds of jobs circulate: kind A and kind B. At a certain node in the network

we have two parallel devices, D_1 and D_2. Both A and B prefer D_1 over D_2, and D_1 prefers A over B, i.e. if D_1 is working on a job of kind B, and a job of kind A arrives, then D_1 immediatly starts working on A, and B is sent to device D_2, if this one is idle; to a queue otherwise (so-called preemptive piority). So, if the events "finish job B on server D_1" and "arrival of job A" are both scheduled, and the latter is scheduled before the former, the former won't happen at all.

The set $\{t_k(e) | e \in E(x_{k-1})\}$ can be viewed as a calendar. We demand that all $t_k(e) > \Theta_{k-1}$.

The event that will actually take place, is the one with the smallest $t_k(e)$. This event is the one that we called e_k, and it terminates state x_{k-1}. This happens at point of time Θ_k, as we saw before; so $\Theta_k = t_k(e_k)$. Then the system enters state x_k, which is completely determined by x_{k-1} and e_k. By doing this step by step, we know the progress of the system essentially, provided that we know the "calenders" $\{t_1(e) | e \in E(x_0)\}$, $\{t_2(e) | e \in E(x_1)\}$, etc.

We assume that when the system is initialized in state x_0, then in one way or another, we know $\{t_1(e) | e \in E(x_0)\}$. Now if we can make an induction step from state x_{k-1} to state x_k, then we have a complete description of the simulation.

At point of time Θ_k the system leaves x_{k-1} because of $e_k \in E(x_{k-1})$. The system enters state x_k, in which the possible events are collected in the set $E(x_k)$. Consider any $e \in E(x_k)$. What will be $t_{k+1}(e)$, the point of time connected with e? We restrict ourselves to two possibilities:

(i) there is some $e' \in E(x_{k-1})$, $e' \neq e_k$, so that $t_{k+1}(e) = t_k(e')$ (1.3.a)

(ii) $t_{k+1}(e) = \Theta_k +$ (some random number, to be discussed presently). (1.3.b)

Example. In the M/M/1-queue we have, except for the idle state, two events in $E(x)$. One of these will take place; the other one will maintain its scheduled time; so we use method (i). However, we might as well use method (ii), as the process is memoryless. If an arrival has taken place, we must use method (ii) for the next arrival.

Next consider the example with the two kinds of jobs A and B, and parallel devices D_1 and D_2. If we assume that device D_2 "finishes the job" if an interrupted job B comes from D_1, then the event "finish B on D_2" will have the same point of time as the former event "finish B on D_1", so rule (i) is used with $e' \neq e$; if D_2 has to start all over again, rule (ii) is appropriate.

How do we find $t_{k+1}(e)$, when method (ii) is used? In this case

$$t_{k+1}(e) = \Theta_k + g$$

where g is a continuous positive random variable, with a distribution depending on x_{k-1} (the latest state the system was in), e_k (the latest event), and e (the event of interest). So we write explicitly

$$t_{k+1}(e) = \Theta_k + g_{x_{k-1},e_k,e} \cdot \qquad (1.4)$$

To assure the uniqueness of e_k, there must be exactly one $e \in E(x_{k-1})$ such that $t_k(e)=\Theta_k$. The way in which $t_k(e)$ is computed, guarantees this with probability 1, provided that if two different events have associated times computed by rule (1.3.a), then the events from which their times originate, are also different; in symbols:

be $e,e^* \in E(x_k)$, $e \neq e^*$, and let there be $e',e^{*\prime} \in E(x_{k-1})$, such that

$t_{k+1}(e)=t_k(e')$, $t_{k+1}(e^*)=t_k(e^{*\prime})$.

Then $e' \neq e^*$'.

An alternative formulation is:

let $e' \in E(x_{k-1})$, $e' \neq e_k$. Then there is at most one $e \in E(x_k)$ so that $t_{k+1}(e)=t_k(e')$.

Note. Instead of methods (i) and (ii), we might have given a more general way of computation, such as

there is some $e' \in E(x_{k-1})$, $e' \neq e$, such that

$t_{k+1}(e) = \Theta_k + f(g, t_k(e')-\Theta_k)$

in which g is some random number and f is some function.

In this paper we have disregarded this kind of generalization, as it would make the sequal far more complicated without clarifying anything more.

2. Derivates of random variables.

In order to discuss P/A we need derivatives of random variables, the topic of this section. Suppose g is a positive continuous random variable (such as time to the next event) with a distribution depending on some parameter ν. For $\nu=\nu_0$, g has cumulative distribution F_{ν_0}. During the simulation we obtain observations g_0 of g by generating (pseudo-)random numbers u from a uniform distribution on (0,1), and solving $u=F_{\nu_0}(g_0)$, that is we use the inverse transformation

$$g_0 = F^{-1}_{\nu_0}(u). \qquad (2.1)$$

If we change ν from ν_0 to $\nu_0+\Delta\nu$, and keep u constant (i.e. we use common random numbers), then the observation will change into

$$g_0 + \Delta g = F^{-1}_{\nu_0+\Delta\nu}(u) = F^{-1}_{\nu_0+\Delta\nu}(F_{\nu_0}(g_0)). \tag{2.2}$$

See Suri (1983) for a further discussion on this subject.

Instead of $u=F_\nu(g)$ we can write $u=F(\nu,g)$. If F, as a function of ν and g, is differentiable to both ν and g, then we can differentiate the latter equation to ν, keeping u constant. This yields

$$0 = \frac{\partial F}{\partial \nu} + \frac{dF}{dg}\frac{dg}{d\nu} \tag{2.3.a}$$

or

$$\frac{dg}{d\nu} = -\left[\frac{dF}{dg}\right]^{-1}\frac{\partial F}{\partial \nu}. \tag{2.3.b}$$

Example. When customers arrive in a system according to an exponential distribution with parameter ν, the cumulative distribution function is $F_\nu(g)=1-e^{-\nu g}$. Equations (2.3) show that

$$0 = ge^{-\nu g} + \nu e^{-\nu g}\frac{dg}{d\nu} \tag{2.4.a}$$

or

$$\frac{dg}{d\nu} = -(\nu e^{-\nu g})^{-1} g e^{-\nu g} = -g/\nu. \tag{2.4.b}$$

In a simulation, a realisation of a random variable g from an exponential distribution is computed from a value u, sampled from a uniform distribution on (0,1), as $g = -\nu^{-1}\,\ell n\, u$. Hence

$$\frac{dg}{d\nu} = \nu^{-2}\,\ell n\, u = -g/\nu \tag{2.5}$$

which is identical to eq. (2.4.b). We interpret (2.5) as follows: the marginal effect of the parameter ν is higher, the higher g is; see figure 1.

FIGURE 1 ABOUT HERE

Example. When customers arrive in a system according to an exponential distribution with parameter ν, the arrival-time of the k^{th} customer is the arrival-time of the $(k-1)^{st}$ customer, plus a random variable g_k, with cumulative distribution function (cdf) $F=1-e^{-\nu g_k}$. In symbols

$$t_k = t_{k-1} + g_k \tag{2.6}$$

or $t_k(\nu)=t_{k-1}(\nu)+g_k(\nu)$ in which we can take $t_0=0$. The derivative of t_k to ν is now simply

$$\frac{dt_k}{d\nu} = \frac{dt_{k-1}}{d\nu} + \frac{dg_k}{d\nu} \tag{2.7}$$

in which $\frac{dg_k}{d\nu}$ is computed as in the former example. We might also take a look at the cdf of t_k. It is simply verified that this cdf is 0 if $t_k \leq t_{k-1}$; and if $t_k \geq t_{k-1}$ then it is

$$1 - e^{-\nu(t_k - t_{k-1})}. \tag{2.8}$$

Again we take $u=1-e^{-\nu(t_k-t_{k-1})}$ and we differentiate to ν, keeping u constant. This yields

$$0 = t_k - t_{k-1} + \nu \frac{dt_k}{d\nu} - \nu \frac{dt_{k-1}}{d\nu} .$$

So, using $g_k=t_k-t_{k-1}$ (2.6) and $\frac{dg_k}{d\nu}=-g_k/\nu$ (2.5), we get (2.7), as is easily verified. This example shows that, if the distribution of a variable depends on ν and some other parameters, these other parameters have to be taken into account in the computation of the derivative.

If g is a random variable with a distribution depending on ν and some other parameters, collected in a vector a with elements also dependent on ν, then the former definitions need some adjustment. Suppose that

$$a_0 = a(\nu_0)$$

$$a_0 + \Delta a = a(\nu_0+\Delta\nu)$$

and g has cumulative distribution $F_{a,\nu}$. Then (2.2) changes into

$$g_0 + \Delta g = F^{-1}_{a_0+\Delta a,\nu_0+\Delta\nu}(F_{a_0,\nu_0}(g_0)) \qquad (2.9)$$

and, by writing $u=F(a,\nu;g)$, differentiation with respect to ν, keeping u constant yields the analogs of (2.3):

$$0 = \sum_j \frac{dF}{da_j}\frac{da_j}{d\nu} + \frac{\partial F}{\partial \nu} + \frac{dF}{dg}\frac{dg}{d\nu} \qquad (2.10.a)$$

$$\frac{dg}{d\nu} = -\left[\frac{dF}{dg}\right]^{-1}\left[\sum_j \frac{dF}{da_j}\frac{da_j}{d\nu} + \frac{\partial F}{\partial \nu}\right] . \qquad (2.10.b)$$

3. Perturbed systems.

We now consider the DEDS (see section 1) not only for $\nu=\nu_0$ but also for $\nu=\nu_0+\Delta\nu$ (cf. section 2). The system description we gave in section 1 is valid for both $\nu=\nu_0$ and $\nu=\nu_0+\Delta\nu$, but all variables introduced in section 1 are dependent on the value of ν, so we should write $x_k(\nu_0)$ or $x_k(\nu_0+\Delta\nu)$ instead of x_k, etc. From now on, if we write x_k (y_k,Θ_k, etc.), we always mean $x_k(\nu_0)$, etc. If $\nu\neq\nu_0$, this will be explicitly stated.

The state of the system, the events and the other variables describing the system, are not only dependent on ν, but also on a random number stream. If we call this random number stream $\Omega=(\omega_1,\omega_2,\ldots)'$, then the correct notation for state k is $x_k(\nu_0,\Omega)$. We omit Ω if this cannot cause confusion.

Suppose z is a function of ν. Then we write Δz short for $z(\nu_0+\Delta\nu)-z(\nu_0)$, and if z is a differentiable function of ν, then dz is short for $\Delta\nu\frac{dz}{d\nu}(\nu_0)$.

Suppose we know the history of the system up to the point where it enters $x_k(\nu_0)$ and $x_k(\nu_0+\Delta\nu)$ respectively. We now investigate what happens in one step of the system, i.e. until the system reaches $x_{k+1}(\nu_0)$ and $x_{k+1}(\nu_0+\Delta\nu)$. By doing so, we can afterwards use induction arguments to describe the simulated system and the perturbations as a whole.

Assume that up till entering x_k the change of ν from ν_0 to $\nu_0+\Delta\nu$ does not effect the state, so $x_k(\nu_0+\Delta\nu) = x_k(\nu_0)$. Then the set of possible events is not changed by the change in ν, so $E(x_k(\nu_0+\Delta\nu)) = E(x_k(\nu_0))$. However, the time at which we entered x_k does change: we entered x_k at $\Theta_k(\nu_0)$ or at $\Theta_k(\nu_0+\Delta\nu)$, for which we assume that

$$\Theta_k(\nu_0+\Delta\nu) = \Theta_k(\nu_0) + d\Theta_k + o(\Delta\nu) \qquad (3.1)$$

in which the symbol $o(\Delta\nu)$ means that that the random variable

$$(\Theta_k(\nu_0+\Delta\nu) - \Theta_k(\nu_0) - d\Theta_k)(\Delta\nu)^{-1}$$

has limit 0 for $\Delta\nu\to 0$ for any fixed value ν_0 of ν and for any given $\Theta_k(\nu_0)$.

The times $t(e)$ can also change. We assume that for any $e\in E(x_k)$

$$t_{k+1}(e,\nu_0+\Delta\nu) = t_{k+1}(e,\nu_0) + dt_{k+1}(e) + o(\Delta\nu). \tag{3.2}$$

Now $\Theta_{k+1}(\nu_0) = \min\{t_{k+1}(e,\nu_0) \mid e\in E(x_k)\}$. For $\Delta\nu$ sufficiently small, $t_{k+1}(e,\nu_0) < t_{k+1}(e',\nu_0)$ implies $t_{k+1}(e,\nu_0+\Delta\nu) < t_{k+1}(e',\nu_0+\Delta\nu)$, so e_{k+1}, the event with the smallest t_{k+1} for $\nu=\nu_0$, is also the event with the smallest t_{k+1} for $\nu=\nu_0+\Delta\nu$, which means that $e_{k+1}(\nu_0+\Delta\nu) = e_{k+1}(\nu_0)$. As

$$\begin{aligned}\Theta_{k+1}(\nu_0+\Delta\nu) &= t_{k+1}(e_{k+1},\nu_0+\Delta\nu)\\ &= t_{k+1}(e_{k+1},\nu_0) + dt_{k+1}(e_{k+1}) + o(\Delta\nu)\\ &= \Theta_{k+1}(\nu_0) + dt_{k+1}(e_{k+1}) + o(\Delta\nu)\end{aligned}$$

we see that, if we take $d\Theta_{k+1} = dt_{k+1}(e_{k+1})$, we get

$$\Theta_{k+1}(\nu_0+\Delta\nu) = \Theta_{k+1}(\nu_0) + d\Theta_{k+1} + o(\Delta\nu). \tag{3.3}$$

We assumed $x_k(\nu_0+\Delta\nu) = x_k(\nu_0)$ and saw that $e_{k+1}(\nu_0+\Delta\nu) = e_{k+1}(\nu_0)$. We conclude that $x_{k+1}(\nu_0+\Delta\nu) = x_{k+1}(\nu_0)$. Again the set of possible events does not change, but how will the times $t_{k+2}(e)$, $e\in E(x_{k+1})$, change?

Be $e\in E(x_{k+1})$. Then either (i) $t_{k+2}(e) = t_{k+1}(e')$ for some $e'\in E(x_k)$, or (ii) $t_{k+2}(e) = \Theta_{k+1} + g_{x_k,e_{k+1},e}$, see (1.3) and (1.4). If (i) holds, then

$$t_{k+2}(e,\nu_0+\Delta\nu) = t_{k+1}(e',\nu_0+\Delta\nu)$$

$$= t_{k+1}(e',\nu_0) + dt_{k+1}(e') + o(\Delta\nu)$$
$$= t_{k+2}(e,\nu_0) + dt_{k+1}(e') + o(\Delta\nu)$$

and we see that, if we take $dt_{k+2}(e) = dt_{k+1}(e')$, then

$$t_{k+2}(e,\nu_0+\Delta\nu) = t_{k+2}(e,\nu_0) + dt_{k+2}(e) + o(\Delta\nu). \qquad (3.4)$$

If (ii) holds, then

$$t_{k+2}(e,\nu_0+\Delta\nu) = \Theta_{k+1}(\nu_0+\Delta\nu) + g_{x_k,e_{k+1},e}(\nu_0+\Delta\nu) \qquad (3.5)$$

in which

$$\Theta_{k+1}(\nu_0+\Delta\nu) = \Theta_{k+1}(\nu_0) + d\Theta_{k+1} + o(\Delta\nu) \qquad (3.6)$$

according to (3.3), and

$$g_{x_k,e_{k+1},e}(\nu_0+\Delta\nu) = F^{-1}_{x_k,e_{k+1},e,\nu_0+\Delta\nu}(F_{x_k,e_{k+1},e,\nu_0}(g_{x_k,e_{k+1},e}(\nu_0)))$$

where F_{x_k,e_{k+1},e,ν_0} is the cdf of $g_{x_k,e_{k+1},e}(\nu_0)$; an analoguous definition holds for $\nu_0+\Delta\nu$. For simplicity's sake, we omit the indices x_{k+1}, e_{k+1} and e, and we rewrite

$$g(\nu_0+\Delta\nu) = F^{-1}_{\nu_0+\Delta\nu}(F_{\nu_0}(g(\nu_0)))$$

which is the same as

$$F_{\nu_0+\Delta\nu}(g(\nu_0+\Delta\nu)) = F_{\nu_0}(g(\nu_0)).$$

This relation remains true, if we write the indices as arguments:

$$F(g(\nu_0+\Delta\nu),\nu_0+\Delta\nu) = F(g(\nu_0),\nu_0). \tag{3.7}$$

We define (using the notation ":=" for "is defined as")

$$\Delta g := g(\nu_0+\Delta\nu) - g(\nu_0) \tag{3.8}$$

and write out

$$\begin{aligned} F(g(\nu_0+\Delta\nu),\nu_0+\Delta\nu) &= F(g(\nu_0)+\Delta g,\nu_0+\Delta\nu) \\ &= F(g(\nu_0),\nu_0) + \Delta g\frac{dF}{dg} + \Delta\nu\frac{\partial F}{\partial\nu} + \\ &\quad + \Delta g\Delta\nu\frac{\partial^2 F}{\partial g\partial\nu} + o(\Delta g) + o(\Delta\nu). \end{aligned}$$

This is substituted into (3.7), yielding

$$\Delta g(\frac{dF}{dg} + \Delta\nu\frac{\partial^2 F}{\partial g\partial\nu} + o(1)) = -\Delta\nu\frac{\partial F}{\partial\nu} + o(\Delta\nu)$$

or, after some elementary algebra,

$$\begin{aligned} \Delta g &= -\Delta\nu(\frac{dF}{dg} + \Delta\nu\frac{\partial\ F}{\partial g\partial\nu} + o(1))^{-1}(\frac{\partial F}{\partial\nu} + o(\Delta\nu)) \\ &= \Delta\nu\frac{dg}{d\nu} + o(\Delta\nu). \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we conclude

$$g(\nu_0+\Delta\nu) = g(\nu_0) + dg + o(\Delta\nu). \tag{3.10}$$

Substituting (3.6) and (3.10) into (3.5), we obtain, with $dt_{k+2}(e)=d\Theta_{k+1}+dg$, that

$$t_{k+2}(e,\nu_0+\Delta\nu) = t_{k+2}(e,\nu_0) + dt_{k+2}(e) + o(\Delta\nu). \tag{3.11}$$

Resuming: there are two ways to compute $t_{k+2}(e)$. In both ways the relation (3.2) is carried over one interval. The assumption (3.1) is also carried over one interval, and so does the assumption $x(\nu_0+\Delta\nu) = x(\nu_0)$. So if we start the simulation in $x_0(\nu_0+\Delta\nu)=x_0(\nu_0)$ at point of time $\Theta_0(\nu_0+\Delta\nu)=\Theta_0(\nu_0)=0$ and we make sure that $t_1(e,\nu_0+\Delta\nu)=t_1(e,\nu_0)+dt_1(e)+o(\Delta\nu)$ for all $e\in E(x_0)$, then induction leads to the conclusion that for any $k=0,1,2,.....$

$$x_k(\nu_0+\Delta\nu) = x_k(\nu_0) \tag{3.12.a}$$

$$\Theta_{k+1}(\nu_0+\Delta\nu) = \Theta_{k+1}(\nu_0) + d\Theta_{k+1} + o(\Delta\nu) \tag{3.12.b}$$

$$t_{k+1}(e,\nu_0+\Delta\nu)=t_{k+1}(e,\nu_0)+dt_{k+1}(e)+o(\Delta\nu),\ \forall e\in E(x_k). \tag{3.12.c}$$

Can we derive relations like (3.12) for the PM's too? A basic PM was defined as

$$y_k^b = (\Theta_k-\Theta_{k-1})f_s(x_{k-1}) + f_o(e_k).$$

If we change ν from ν_0 to $\nu_0+\Delta\nu$, the x_{k-1} and e_k do not change; only the Θ's do. We have

$$\begin{aligned} y_k^b(\nu_0+\Delta\nu) &= \{\Theta_k(\nu_0+\Delta\nu) - \Theta_{k-1}(\nu_0+\Delta\nu)\}f_s(x_{k-1}) + f_o(e_k) \\ &= \{\Theta_k(\nu_0)-\Theta_{k-1}(\nu_0)+d\Theta_k-d\Theta_{k-1}+o(\Delta\nu)\}f_s(x_{k-1})+f_o(e_k) \\ &= y_k^b(\nu_0) + (d\Theta_k-d\Theta_{k-1})f_s(x_{k-1}) + o(\Delta\nu). \end{aligned} \tag{3.13}$$

For a simple PM it immediately follows from

$$y_{K_1,K_2} = \frac{1}{K_2-K_1+1}\sum_{k=K_1}^{K_2} y_k^b$$

that

$$y_{K_1,K_2}(\nu_0+\Delta\nu)=y_{K_1,K_2}(\nu_0)+\frac{1}{K_2-K_1+1}\Sigma_{k=K_1}^{K_2}(d\Theta_k-d\Theta_{k-1})f_s(x_{k-1})+o(\Delta\nu),$$

so that

$$\frac{dy_{K_1,K_2}}{d\nu} = \frac{1}{K_2-K_1+1}\Sigma_{k=K_1}^{K_2}(\frac{d\Theta_k}{d\nu} - \frac{d\Theta_{k-1}}{d\nu})f_s(x_{k-1}). \qquad (3.14)$$

So to compute the derivative of a simple PM to ν, the only information we need is the derivative of the points of time Θ_k. The latter follow from the derivatives of the scheduled points of time $t_k(e)$. Only if $t_k(e)$ is computed according to (1.3.b), then we need an extra computation, more complicated than an addition. The derivatives are computed in the same simulation run as the PM's themselves.

If $\Delta\nu$ is small enough - and we do take $\Delta\nu$ infinitesimally small - the history of the simulation is the same, whether $\nu=\nu_0$ or $\nu=\nu_0+\Delta\nu$. The only things that change, are the lengths of the successive intervals, as reflected in (3.14), where we see that a simple PM is a differentiable function of ν. A general PM was defined as a function of simple PM's. Now we define an admissible PM as a differentiable function of simple PM's, which immediately gives as a result: an admissible PM is differentiable to ν.

4. Discussion.

We have given a description of a DEDS, and we have seen how P/A can be explained in our terminology. The merits of P/A are not further discussed in this article; see Heidelberger (1987) for the most recent contribution to this discussion.

The description presented here, is very general in that it is not limited to a certain simulation language. It can be extented to more situations that can occur in a DEDS. We give the following examples.

Simultaneity. Imagine a situation in which a workpiece, upon leaving some device, has probability p of joining one queue, and probability q =

1-p of joining another queue. This kind of situations does not fit into our discription, in which we demanded that a state and an event together completely determine the successor state. We might say that, instead of the event "finish at A", followed by either "start at B" or "start at C", we have two events "finish at A, start at B" and "finish at A, start at C", which are scheduled at the same time. But then $\min\{t_k(e) | e \in E(x)\}$ can be assumed by two different events e, which we prohibited! It is clear that the description has to be extended to include this possibility.

Preemption. In systems with preemption it may occur, that a the arrival of a higher-priority workpiece interrupts the work on a lower-priority workpiece. The work on the latter is resumed after finishing the former. While the device is busy with the high priority workpiece, the remaining time needed for the other one must be part of the state. In this case a change in ν will result in a change in the state. This possibility needs a more refined definition of state than used in the present description of a DEDS.

The given description of a DEDS is not yet suited to cover every possible DEDS, but it may prove to be worthwile as a basis for a really general description.

Appendix. A general PM is indeed general.

We want to prove that any PM is a general PM. We introduced a PM as a function of the states and events the system goes through during the simulation, and the duration of the states. So any PM can be written as

$$z(x_0,\Theta_1-\Theta_0,e_1,x_1,\Theta_2-\Theta_1,e_2,\ldots,x_{\ell-1},\Theta_\ell-\Theta_{\ell-1},e_\ell,\ldots,x_{K-1},\Theta_K-\Theta_{K-1},e_K).$$

Both X and E are countable. We define functions $h_s^{(\ell)}:X\to\mathbb{N}$ and $h_o^{(\ell)}:E\to\mathbb{N}$ ($\ell=1,\ldots,K$), and a function $\tilde{z}:\mathbb{R}^{3K}\to\mathbb{R}$, so that

$$\tilde{z}(h_s^{(1)}(x_0),\Theta_1-\Theta_0,h_o^{(1)}(e_1),h_s^{(2)}(x_1),\Theta_2-\Theta_1,h_o^{(2)}(e_2),\ldots\ldots,$$
$$h_s^{(\ell)}(x_{\ell-1}),\Theta_\ell-\Theta_{\ell-1},h_o^{(\ell)}(e_\ell),\ldots\ldots,h_s^{(K)}(x_{K-1}),\Theta_K-\Theta_{K-1},h_o^{(K)}(e_K)) =$$

$$z(x_0,\Theta_1-\Theta_0,e_1,x_1,\Theta_2-\Theta_1,e_2,\ldots,x_{\ell-1},\Theta_\ell-\Theta_{\ell-1},e_\ell,\ldots,x_{K-1},\Theta_K-\Theta_{K-1},e_K)$$

for all values of all arguments. Now we construct PM's as follows. First we define for $\ell=1,\ldots,K$, $x\in X$, $e\in E$ the functions

$$f_s^{(1,\ell)}(x) := h_s^{(\ell)}(x) \qquad f_o^{(1,\ell)}(e) := 0$$
$$f_s^{(2,\ell)}(x) := 1 \qquad f_s^{(2,\ell)}(e) := 0$$
$$f_s^{(3,\ell)}(x) := 0 \qquad f_s^{(3,\ell)}(e) := h_o^{(\ell)}(e).$$

Associated with these are the following basic PM's:

$$y_k^{b(1,\ell)} = (\Theta_k-\Theta_{k-1})h_s^{(\ell)}(x_{k-1})$$
$$y_k^{b(2,\ell)} = \Theta_k-\Theta_{k-1}$$
$$y_k^{b(3,\ell)} = h_o^{(\ell)}(e_k)$$

for any $\ell=1,\ldots,K$. Corresponding with these we define 3K simple PM's

$$y^{(1,\ell)}_{\ell,\ell} = (\Theta_\ell - \Theta_{\ell-1}) h_s^{(\ell)}(x_{k-1})$$
$$y^{(2,\ell)}_{\ell,\ell} = \Theta_\ell - \Theta_{\ell-1}$$
$$y^{(3,\ell)}_{\ell,\ell} = h_o^{(\ell)}(e_\ell),$$

again for $\ell=1,\ldots,K$. Now

$$\tilde{z}(y^{(1,1)}_{1,1}/y^{(2,1)}_{1,1}, y^{(2,1)}_{1,1}, y^{(3,1)}_{1,1}, y^{(2,1)}_{2,2}/y^{(2,2)}_{2,2}, y^{(2,2)}_{2,2}, y^{(3,2)}_{2,2}, \ldots$$
$$\ldots, y^{(1,\ell)}_{\ell,\ell}/y^{(2,\ell)}_{\ell,\ell}, y^{(2,\ell)}_{\ell,\ell}, y^{(3,\ell)}_{\ell,\ell}, \ldots, y^{(1,K)}_{K,K}/y^{(2,K)}_{K,K}, y^{(2,K)}_{K,K}, y^{(3,K)}_{K,K})$$

is a function of simple PM's; so by definition it is a general PM. It equals the arbitrary PM z we started with, so we may conclude that indeed any PM is a general PM.

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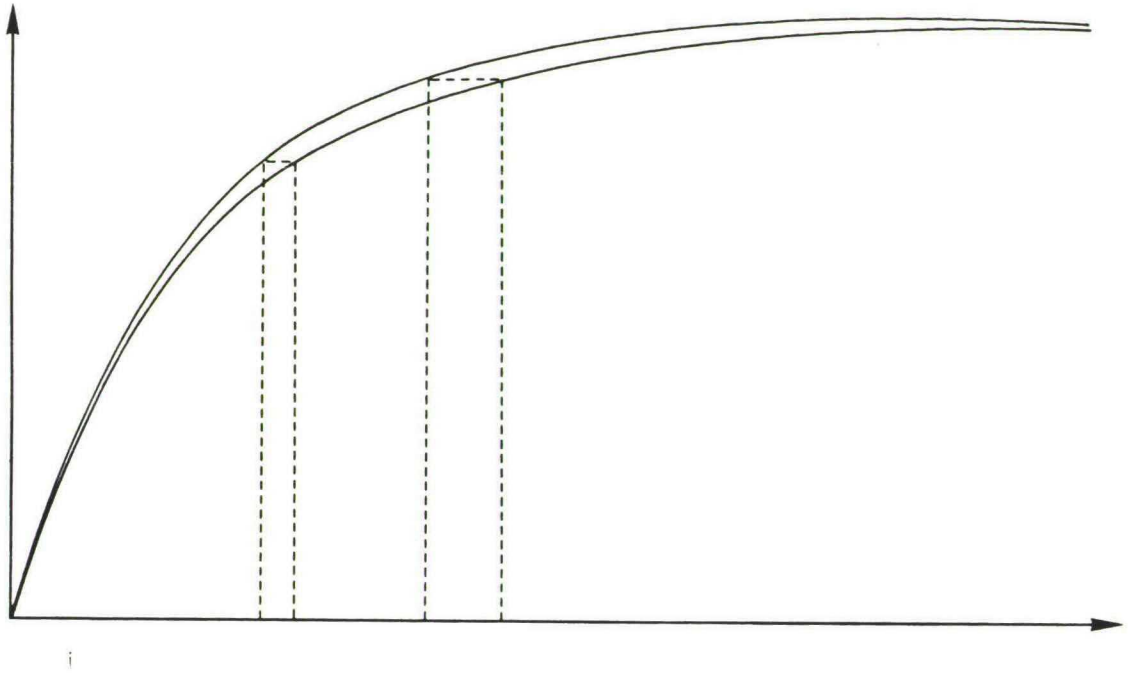

Fig. 1. The effect on g of a change in ν.

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